't Hooft GR exercise 4.1 pg 16.

We'd like to verify $\hat{A}_\nu(u) = x^\mu_{,\nu} A_\mu(x(u))$ as a transformation from $x \to u$ forms a group.

$$\begin{tikzcd} x \arrow[r] \arrow[dr] & v \arrow[d] \\ & u \end{tikzcd}$$

$x \to v$: $\tilde{A}_\nu(v) = x^\mu_{,\nu} A_\mu(x(v))$

$v \to u$: $\tilde{A}_\lambda(u) = v^\nu_{,\lambda} \tilde{A}_\nu(v(u))$

$\Rightarrow x \to v \to u$: $\tilde{A}_\lambda(u) = v^\nu_{,\lambda} x^\mu_{,\nu} A_\mu(x(v))$.

$$= \frac{\partial v^\nu}{\partial u^\lambda} \frac{\partial x^\mu}{\partial v^\nu} \tilde{A}_\mu(x(v))$$

$$= \frac{\partial x^\mu}{\partial u^\lambda} \tilde{A}_\mu(x(v(u)))$$

$$= \boxed{\frac{\partial x^\mu}{\partial u^\lambda} \tilde{A}_\mu(x(u))}$$

$\Updownarrow$

$x \to u$ directly

Now for $\tilde{F}^{\mu}(u) = u^{\mu}{}_{,\alpha} F^{\alpha}(x(u))$ as $x \to u$,

$x \to v$: $\quad \tilde{F}^{\mu}(v) = v^{\mu}{}_{,\alpha} F^{\alpha}(x(v))$

$v \to u$: $\quad \tilde{F}^{\lambda}(u) = u^{\lambda}{}_{,\mu} F^{\mu}(u(v))$

$\Rightarrow x \to v \to u$: $\quad \tilde{F}^{\lambda}(u) = u^{\lambda}{}_{,\mu} v^{\mu}{}_{,\alpha} F^{\alpha}(x(v))$

$$= \frac{\partial u^{\lambda}}{\partial v^{\mu}} \frac{\partial v^{\mu}}{\partial x^{\alpha}} F^{\alpha}(x(v(u)))$$

$$= \boxed{\frac{\partial u^{\lambda}}{\partial x^{\alpha}} F^{\alpha}(x(u))}$$

$$\Updownarrow$$

$x \to u$ directly

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